6.3 – Gram-Schmidt Process

Definition: A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be **orthonormal**.

Theorem 6.3.1 If $S = \{\mathbf{v}_1,\ \mathbf{v}_2, \ldots,\ \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Definition: In an inner product space, a basis comprising orthogonal vectors is an **orthogonal basis**, and a basis comprising orthonormal vectors is an **orthonormal basis**.

#6 Show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space.

$$A = \begin{bmatrix} 1/5 & -2/3 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix}$$

Theorem 6.3.2

a) If $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ldots, \ \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \ \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \ \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots \frac{\langle \mathbf{u}, \ \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

b) If $S = \{\mathbf{v}_1, \ \mathbf{v}_2, \ldots, \ \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then $\mathbf{u} = \langle \mathbf{u}, \ \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \ \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \ \mathbf{v}_n \rangle \mathbf{v}_n$.

#8 Use Theorem 6.3.2(b) to express the vector $\mathbf{u} = (3,\ -7,\ 4)$ as a linear combination of the vectors $\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5},\ 0\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5},\ 0\right)$, $\mathbf{v}_3 = (0,\ 0,\ 1)$.

#12 Find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in Exercise 8.

Theorem 6.3.3 Projection Theorem (generalization of Theorem 3.3.2)
If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.

Definition: If W is a finite-dimensional subspace of an inner product space V and a vector $\mathbf{u}$ in V is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$, then $\mathbf{w}_1$ is called the **orthogonal projection of u on W** and is denoted by $\mathbf{w}_1 = \text{proj}_W \mathbf{u}$ and $\mathbf{w}_2$ is called the **orthogonal projection of u on W**$^{\perp}$ and is denoted by $\mathbf{w}_2 = \text{proj}_{W^{\perp}} \mathbf{u}$. The vector $\mathbf{w}_2$ is also called the **component of u orthogonal to W**.

Theorem 6.3.4 Let W be a finite-dimensional subspace of an inner product space V.
a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \ldots \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r$$

b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

#24 The vectors $\mathbf{v}_1 = (0, 1, -4, -1)$ and $\mathbf{v}_2 = (3, 5, 1, 1)$ are orthogonal with respect to the Euclidean inner product on R^4. Find the orthogonal projection $\mathbf{w}_1$ of $\mathbf{b} = (1, 2, 0, -2)$ on the subspace W spanned by these vectors. [Extension of the exercise] Then find $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

#25 The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -1)$ onto the subspace W spanned by these vectors.

$\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$, $\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$

Theorem 6.3.5 (proof outlines the Gram-Schmidt process)
Every nonzero finite-dimensional inner product space has an orthonormal basis.

#31 Let R^4 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.
$\mathbf{u}_1 = (0, 2, 1, 0)$, $\mathbf{u}_2 = (1, -1, 0, 0)$, $\mathbf{u}_3 = (1, 2, 0, -1)$, $\mathbf{u}_4 = (1, 0, 0, 1)$

Theorem 6.3.6 (inner product space analog of Theorem 4.6.5 (b))
If W is a finite-dimensional inner product space, then:
a) Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.
b) Every orthonormal set of nonzero vectors in W can be enlarged to an orthonormal basis for W.

Example 6.3.9 Legendre Polynomials
Let the vector space P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$. Apply the Gram-Schmidt process to transform the standard basis $\left\{1, x, x^2\right\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.

Definition: The first three **Legendre polynomials** are 1, x, and $\frac{1}{2}\left(3x^2 - 1\right)$. They result from transforming the standard basis for P_2 with the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$ into an orthogonal basis for P_2 via the Gram-Schmidt process, scaled so they have a value of 1 at $x = 1$.